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Reformulation of Possio's Kernel with Application to Unsteady Wind Tunnel Interference

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An efficient method for computing the Possio kernel has remained elusive up to the present time. In this paper we reformulate the Possio kernel so that it can be computed accurately using existing high precision numerical quadrature techniques. Convergence to the correct values is demonstrated and optimization of the integration procedures is discussed. Since more general kernels such as those associated with unsteady flows in ventilated wind tunnels are analytic perturbations of the Possio free air kernel, a more accurate evaluation of their collocation matrices results with an exponential improvement in convergence. An application to predicting frequency response of an airfoil-trailing edge control system in a wind tunnel compared with that in free air is given showing strong interference effects.

I. Introduction

A PROBLEM of long standing concern in unsteady aerodynamics has been the efficient computation of the Possio kernel[1] and the identification of its singularities. In this paper, we present a new expression for the Possio kernel which can be computed efficiently and accurately by well-known techniques.

Since more general problems associated with unsteady interference effects in two-dimensional wind tunnels lead to kernels[2,3] which are analytic perturbations of the free air Possio kernel, our procedure results in more effective methods for computing interference effects. This comes about because the singularities of the kernels for linearized subsonic flow in wind tunnels ventilated by slotted or porous walls are identical to the singularities of the Possio kernel, the separate integration of which leads to an accurate evaluation of the collocation matrices.

In Sec. II, the Possio kernel is recast to display its singularities explicitly and in Sec. III an efficient method is described for its computation. In Sec. IV this result is applied to a problem in unsteady wind tunnel flow where it is shown that an exponential improvement in convergence over the previous solution method occurs.[3,4]

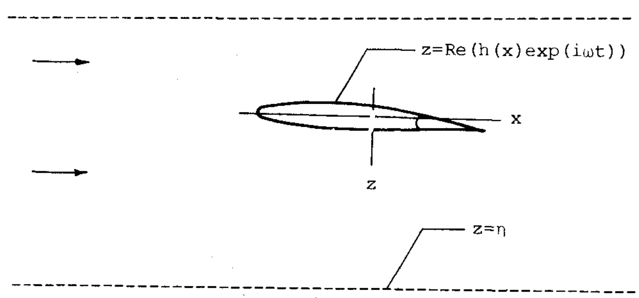

Fig. 1 Coordinate system and sign conventions.

The geometry and coordinate system used are shown in Fig. 1. Following customary practice, all lengths are nondimensionalized by the airfoil semichord b, pressures by the freestream dynamic pressure, and the downwash w by freestream velocity v_∞. The airfoil is assumed to be thin and to be located midway between the tunnel walls. The reduced frequency $k=\omega b/v_\infty$ is based on the semichord, and the co-Mach number is written as $\beta=\sqrt{1-M^2}$. Free air conditions are represented by removing the walls to infinity.

II. Reformulation of Possio Kernel

The Possio kernel[1,3,4] is given by

$$A(x,k,M) = -\frac{\pi k}{2\beta^2}e^{-ikx}\left\{e^{ikx/\beta^2}\left[iM\mathrm{sgn}(x)H_1^{(2)}\left(\frac{Mk|x|}{\beta^2}\right)\right.\right.$$
$$\left.-H_0^{(2)}\left(\frac{Mk|x|}{\beta^2}\right)\right]+\frac{2i\beta}{\pi}\log\frac{1+\beta}{M}$$
$$\left.+ik\int_0^x e^{ik\lambda/\beta^2}H_0^{(2)}\left(\frac{Mk|\lambda|}{\beta^2}\right)\mathrm{d}\lambda\right\} \tag{1}$$

where Hankel functions of the second kind[5] are given in terms of Bessel functions of the first and second kind,

$$H_\ell^{(2)}=J_\ell-iY_\ell \tag{2}$$

Clearly Eq. (1) is not in a form suitable for numerical computation. Its singularities are not explicitly displayed, it is indeterminate for $M=0$, and it contains an integral with variable upper limit. In addition, the appearance of absolute value functions suggests (falsely) streamwise step function singularities at $x=0$.

In this paper we will prove that

$$A(x,k,M)=H(x)+K_1(x,k,M)\log|x|+K_2(x,k,M) \tag{3}$$

where

$$H(x)=1/x \tag{4}$$

is the steady free air kernel, and where

$$K_1(x,k,M)=-(ik/\beta^2)e^{-ikx}F_1(x,k,M) \tag{5}$$

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and

$$K_2(x,k,M) = -e^{-ikx} F_2(x,k,M) \tag{6}$$

are analytic functions of x.

To guide the analysis, we note that the closed form expression for the kernel in the special case of incompressible free air flow[3,4] is

$$A(x,k,0) = \frac{1}{x} - ike^{-ikx}\left[Ci(k|x|) + iSi(kx) + \frac{i\pi}{2}\right] \tag{7}$$

where Ci and Si are the cosine and sine integrals. They are given[5] by

$$Ci(z) = \gamma + \log z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)(2n)!} \tag{8}$$

and

$$Si(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!} \tag{9}$$

where $\gamma = 0.57721566490153...$ is Euler's constant. Combining Eqs. (7-9), we obtain

$$A(x,k,0) = \frac{1}{x} - ike^{-ikx}\log|x| - ike^{-ikx}\left[\log k + \gamma + \frac{i\pi}{2} + \sum_{n=1}^{\infty} \frac{(ikx)^n}{(n)(n!)}\right] \tag{10}$$

Thus we can identify the analytic functions in this case as

$$F_1(x,k,0) = 1 \tag{11}$$

and

$$F_2(x,k,0) = ik\left[\log k + \gamma + \frac{i\pi}{2} + \sum_{n=1}^{\infty} \frac{(ikx)^n}{(n)(n!)}\right] \tag{12}$$

Returning to the problem at hand, substitution of Eq. (2) into Eq. (1) gives

$$A(x,k,M) = e^{-ikx}\Bigg\{ e^{ikx/2\beta^2}\Bigg[-\frac{i\pi Mk}{\beta^2}\operatorname{sgn}(x)\left(J_1\left(\frac{Mk|x|}{\beta^2}\right) - iY_1\left(\frac{Mk|x|}{\beta^2}\right)\right) + \frac{\pi k}{2\beta^2}\left(J_0\left(\frac{Mk|x|}{\beta^2}\right) - iY_0\left(\frac{Mk|x|}{\beta^2}\right)\right)\Bigg] - \frac{ik}{\beta}\log\frac{1+\beta}{M} - \frac{i\pi k^2 x}{2\beta^2}\int_0^1 e^{ikxu/\beta^2}\left[J_0\left(\frac{Mk|x|u}{\beta^2}\right) - iY_0\left(\frac{Mk|x|u}{\beta^2}\right)\right]du\Bigg\} \tag{13}$$

In order to identify the singularities in Eq. (13), we first determine the specific singularities, as well as analytic functions, which appear in the Bessel functions. Clearly

$$J_\ell(z) = \left(\frac{z}{2}\right)^\ell \sum_{m=0}^{\infty} \frac{(-z^2/4)^m}{m!(m+\ell)!} \tag{14}$$

is analytic. Let

$$\psi(1) = -\gamma, \psi(m+1) = \psi(m) + (1/m) \text{ if } m \geq 1 \tag{15}$$

denote values of the psi or digamma function[5] for integer arguments. Then

$$Y_0(z) = \frac{2}{\pi}\left(\gamma + \log\frac{z}{2}\right)J_0(z) + \frac{2}{\pi}G_0(z) \tag{16}$$

$$Y_1(z) = -\frac{2}{\pi z} + \frac{2}{\pi}\log\frac{z}{2}J_1(z) - \frac{2}{\pi}G_1(z) \tag{17}$$

where

$$G_0(z) = \frac{(z/2)^2}{(1!)^2} - \left(1 + \frac{1}{2}\right)\frac{(z/2)^4}{(2!)^2} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{(z/2)^6}{(3!)^2} - \ldots \tag{18}$$

and

$$G_1(z) = \frac{z}{4}\sum_{n=0}^{\infty} [\psi(n+1) + \psi(n+2)]\frac{(-z^2/4)^n}{n!(n+1)!} \tag{19}$$

Obviously G_0 and G_1 and analytic. We note that they may be computed using Eqs. (18) and (19) or, using standard Bessel function subroutines, according to

$$G_0(z) = \frac{\pi}{2}Y_0(z) - \left(\gamma + \log\frac{z}{2}\right)J_0(z) \tag{20}$$

and

$$G_1(z) = -\frac{\pi}{2}Y_1(z) - \frac{1}{z} + \log\frac{z}{2}J_1(z) \tag{21}$$

Combining the above and rearranging, it follows that

$$A(x,k,M) = \frac{1}{x} - \frac{ik}{\beta^2}e^{-ikx}\left[\left(J_0\left(\frac{Mkx}{\beta^2}\right) - iMJ_1\left(\frac{Mkx}{\beta^2}\right)\right)e^{ikx/\beta^2} - ikx\int_0^1 e^{ikxu/\beta^2}J_0\left(\frac{Mkxu}{\beta^2}\right)du\right]\log|x|$$
$$- e^{-ikx}\Bigg\{\Bigg[\frac{e^{iM^2kx/\beta^2} - 1}{x} - \frac{\pi k}{2\beta^2}J_0\left(\frac{Mkx}{\beta^2}\right) + \frac{ik}{\beta^2}G_0\left(\frac{Mkx}{\beta^2}\right) + \frac{ik}{\beta^2}\left(\gamma + \log\frac{Mk}{2\beta^2}\right)J_0\left(\frac{Mkx}{\beta^2}\right) + \frac{i\pi kM}{2\beta^2}J_1\left(\frac{Mkx}{\beta^2}\right)$$
$$+ \frac{Mk}{\beta^2}\log\frac{Mk}{2\beta^2}J_1\left(\frac{Mkx}{\beta^2}\right) - \frac{Mk}{\beta^2}G_1\left(\frac{Mkx}{\beta^2}\right)\Bigg]e^{ikx/\beta^2} + \frac{ik}{\beta}\log\frac{1+\beta}{M} + \frac{i\pi k^2 x}{2\beta^2}\int_0^1 e^{ikxu/\beta^2}\left(J_0\left(\frac{Mkxu}{\beta^2}\right)\right.$$
$$\left. - \frac{2i}{\pi}G_0\left(\frac{Mkxu}{\beta^2}\right)\right)du + \frac{k^2 x}{\beta^2}\left(\gamma + \log\frac{Mk}{2\beta^2}\right)\int_0^1 e^{ikxu/\beta^2}J_0\left(\frac{Mkxu}{\beta^2}\right)du - \frac{k^2 x}{\beta^2}\int_0^1 \log\frac{1}{u}e^{ikxu/\beta^2}J_0\left(\frac{Mkxu}{\beta^2}\right)du\Bigg\} \tag{22}$$

which in view of Eq. (3) gives

$$F_1(x,k,M) = \left[J_0\left(\frac{Mkx}{\beta^2}\right) - iMJ_1\left(\frac{Mkx}{\beta^2}\right)\right]e^{ikx/\beta^2} - ikx\int_0^1 e^{ikxu/\beta^2}J_0\left(\frac{Mkxu}{\beta^2}\right)du \tag{23}$$

and

$$F_2(x,k,M)=\Big[\frac{e^{-iM^2kx/\beta^2}-1}{x}-\frac{\pi k}{2\beta^2}J_0\Big(\frac{Mkx}{\beta^2}\Big)+\frac{ik}{\beta^2}G_0\Big(\frac{Mkx}{\beta^2}\Big)+\frac{ik}{\beta^2}\Big(\gamma+\log\frac{Mk}{2\beta^2}\Big)J_0\Big(\frac{Mkx}{\beta^2}\Big)+\frac{i\pi Mk}{2\beta^2}J_1\Big(\frac{Mkx}{\beta^2}\Big)$$
$$+\frac{Mk}{\beta^2}\log\frac{Mk}{2\beta^2}J_1\Big(\frac{Mkx}{\beta^2}\Big)-\frac{Mk}{\beta^2}G_1\Big(\frac{Mkx}{\beta^2}\Big)\Big]e^{ikx/\beta^2}+\frac{ik}{\beta}\log\frac{1+\beta}{M}+\frac{i\pi k^2x}{2\beta^2}\int_0^1 e^{ikxu/\beta^2}\Big(J_0\Big(\frac{Mkxu}{\beta^2}\Big)-\frac{2i}{\pi}G_0\Big(\frac{Mkxu}{\beta^2}\Big)\Big)\mathrm{d}u$$
$$+\frac{k^2x}{\beta^2}\Big(\gamma+\log\frac{Mk}{2\beta^2}\Big)\int_0^1 e^{ikxu/\beta^2}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u-\frac{k^2x}{\beta^2}\int_0^1\log\frac{1}{u}e^{ikxu/\beta^2}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u \tag{24}$$

Equations (23) and (24) display the desired functions F_1 and F_2, but F_2 is still indeterminate when $M=0$ and it remains to show that they reduce to Eqs. (11) and (12) when $M=0$. F_1 can be put into final form by first noting that

$$J_0'=-J_1 \tag{25}$$

and integrating by parts to yield

$$ikx\int_0^1 e^{ikxu/\beta^2}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u=\beta^2\Big(J_0\Big(\frac{Mkx}{\beta^2}\Big)e^{ikx/\beta^2}-1\Big)+Mkx\int_0^1 e^{ikxu/\beta^2}J_1\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u \tag{26}$$

Substituting Eq. (26) into Eq. (23) and simplifying, we obtain

$$F_1(x,k,M)=1+M^2\Big(J_0\Big(\frac{Mkx}{\beta^2}\Big)e^{ikx/\beta^2}-1\Big)-iMJ_1\Big(\frac{Mkx}{\beta^2}\Big)e^{ikx/\beta^2}-Mkx\int_0^1 e^{ikxu/\beta^2}J_1\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u \tag{27}$$

which clearly is analytic as asserted previously, and reduces upon inspection to Eq. (11) when $M=0$.
A similar reduction may be obtained for F_2. Integrating the last term in Eq. (24) by parts gives

$$-\frac{k^2x}{\beta^2}\int_0^1\log\frac{1}{u}e^{ikxu/\beta^2}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u=ik\int_0^1\frac{e^{ikxu/\beta^2}-1}{u}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u+\frac{iMk^2x}{\beta^2}\int_0^1\log\frac{1}{u}\,(e^{ikxu/\beta^2}-1)J_1\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u \tag{28}$$

[Observe that when $M=0$, Eq. (28) reduces to

$$ik\int_0^1\frac{e^{ikxu}-1}{u}\mathrm{d}u=ik\sum_{n=1}^{\infty}\frac{(ikx)^n}{(n)(n!)} \tag{29}$$

which is the last term in Eq. (12)]. Substituting Eq. (28) into Eq. (24) gives

$$F_2(x,k,M)=ik\int_0^1\frac{e^{ikxu/\beta^2}-1}{u}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u+\frac{iMk^2x}{\beta^2}\int_0^1\log\frac{1}{u}\,(e^{ikxu/\beta^2}-1)J_1\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u$$
$$+ik\Big[\frac{e^{-iM^2kx/\beta^2}-1}{ikx}+\frac{i\pi}{2\beta^2}J_0\Big(\frac{Mkx}{\beta^2}\Big)+\frac{1}{\beta^2}G_0\Big(\frac{Mkx}{\beta^2}\Big)+\frac{1}{\beta^2}\Big(\gamma+\log\frac{Mk}{2\beta^2}\Big)J_0\Big(\frac{Mkx}{\beta^2}\Big)+\frac{\pi M}{2\beta^2}J_1\Big(\frac{Mkx}{\beta^2}\Big)$$
$$-\frac{iM}{\beta^2}\log\frac{Mk}{2\beta^2}J_1\Big(\frac{Mkx}{\beta^2}\Big)+\frac{iM}{\beta^2}G_1\Big(\frac{Mkx}{\beta^2}\Big)\Big]e^{ikx/\beta^2}+\frac{ik}{\beta}\log\frac{1+\beta}{M}+\frac{i\pi k^2x}{2\beta^2}\int_0^1 e^{ikxu/\beta^2}\Big(J_0\Big(\frac{Mkxu}{\beta^2}\Big)$$
$$-\frac{2i}{\pi}G_0\Big(\frac{Mkxu}{\beta^2}\Big)\Big)\mathrm{d}u+\frac{k^2x}{\beta^2}\Big(\gamma+\log\frac{Mk}{2\beta^2}\Big)\int_0^1 e^{ikxu/\beta^2}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u \tag{30}$$

Integrating by parts again gives

$$J_0\Big(\frac{kMx}{\beta^2}\Big)e^{ikx/\beta^2}-ikx\int_0^1 e^{ikxu/\beta^2}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u=\beta^2+M^2J_0\Big(\frac{Mkx}{\beta^2}\Big)e^{ikx/\beta^2}-Mkx\int_0^1 e^{ikxu/\beta^2}J_1\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u \tag{31}$$

Substituting Eq. (31) and the identity

$$ik\Big(\gamma+\log\frac{Mk}{2\beta^2}\Big)-\frac{\pi k}{2}+\frac{ik}{\beta}\log\frac{1+\beta}{M}=ik\Big(\log k+\gamma+\frac{i\pi}{2}\Big)+\frac{ik}{\beta}\Big(\frac{M^2}{1+\beta}\log\frac{2\beta^2}{M}+\log\frac{1+\beta}{2\beta^2}\Big) \tag{32}$$

into Eq. (30), we obtain

$$F_2(x,k,M)=ik\Big[\log k+\gamma+\frac{i\pi}{2}+\int_0^1\frac{e^{ikxu/\beta^2}-1}{u}J_0\Big(\frac{Mkxu}{\beta^2}\Big)\mathrm{d}u\Big]+ik\Big[\frac{e^{-ikM^2x/\beta^2}-1}{ikx}+\frac{M^2}{\beta^2}\Big(\frac{i\pi}{2}+\gamma+\log\frac{Mk}{2\beta^2}\Big)J_0\Big(\frac{Mkx}{\beta^2}\Big)$$
$$-\frac{iM}{\beta^2}\Big(\frac{i\pi}{2}+\log\frac{Mk}{2\beta^2}\Big)J_1\Big(\frac{Mkx}{\beta^2}\Big)+\frac{1}{\beta^2}G_0\Big(\frac{Mkx}{\beta^2}\Big)+\frac{iM}{\beta^2}G_1\Big(\frac{Mkx}{\beta^2}\Big)\Big]e^{ikx/\beta^2}-\frac{ik^2x}{\beta^2}\int_0^1\Big[M\Big(\gamma+\log\frac{Mk}{2\beta^2}+\frac{i\pi}{2}\Big)J_1\Big(\frac{Mkxu}{\beta^2}\Big)$$

$$+iG_0\left(\frac{Mkxu}{\beta^2}\right)\Big]e^{ikxu/\beta^2}\mathrm{d}u+\frac{iMk^2x}{\beta^2}\int_0^1\log\frac{1}{u}J_1\left(\frac{Mkxu}{\beta^2}\right)(e^{ikxu/\beta^2}-1)\mathrm{d}u+\frac{ik}{\beta}\left(\frac{M^2}{1+\beta}\log\frac{2\beta^2}{M}+\log\frac{1+\beta}{2\beta^2}\right) \tag{33}$$

which, in view of Eq. (29), reduces by inspection to Eq. (12) when $M=0$, and is clearly analytic in x, thereby establishing Eq. (3).

III. Efficient Computation of K_1 and K_2

The efficient computation of K_1 and K_2 reduces to that of the four integrals appearing in Eqs. (27) and (33):

$$I_1(M,\tilde{k}x)=\int_0^1\exp(i\tilde{k}xu)J_1(M\tilde{k}xu)\mathrm{d}u \tag{34}$$

$$I_2(M,\tilde{k}x)=\int_0^1\frac{\exp(i\tilde{k}xu)-1}{u}J_0(M\tilde{k}xu)\mathrm{d}u \tag{35}$$

$$I_3(M,\tilde{k}x)=\int_0^1\exp(i\tilde{k}xu)G_0(M\tilde{k}xu)\mathrm{d}u \tag{36}$$

$$I_4(M,\tilde{k}x)=\int_0^1\log\frac{1}{u}[\exp(i\tilde{k}xu)-1]J_1(M\tilde{k}xu)\mathrm{d}u \tag{37}$$

where $\tilde{k}\equiv k/\beta^2$. The first three can be computed by Legendre-Gaussian quadrature and the fourth by logarithmic-Gaussian quadrature. Let I denote the converged value of any one of the integrals above and let I^{NQ} denote its quadrature approximation using NQ nodes. Then the relative error is

$$\epsilon=(I^{NQ}-I)/I \tag{38}$$

and the number of decimals of accuracy is

$$ND=\log_{10}1/|\epsilon| \tag{39}$$

Figure 2 illustrates the computational convergence to I_1 obtained for various values of Mach number, frequency, and distance. The number of decimals of accuracy achieved becomes nearly linear with the number of quadrature nodes so that a small number of function evaluations produces excellent accuracy. The convergence behavior of the remaining three integrals is similar to that of I_1.

In order to compute the integrals I_1-I_4 to within a prescribed accuracy using minimum or near minimum computing time, it is necessary to have a priori knowledge of the number of quadrature nodes required,

$$NQ=f(M,\tilde{k}x,ND) \tag{40}$$

in terms of Mach number, frequency, distance, and the number of decimals of accuracy desired. By examining a large number of graphs such as Fig. 2, we have found the following approximation to be satisfactory,

$$NQ\approx(c_1+c_2M)+(c_3+c_4M)ND+[(c_5+c_6M)+(c_7+c_8M)ND]\tilde{k}x \tag{41}$$

where the coefficients c_i are determined empirically for each of I_1-I_4. Figure 3 depicts the quadrature envelopes so obtained, shown by dashed lines, for $\tilde{k}x=10$. In general, the quadrature envelopes of Eq. (41) are slightly conservative, producing somewhat more accuracy than specified. This is due in part to their being generally upper bounds for NQ, and in part to roundup to the next integer. For smaller values of $\tilde{k}x$, greater conservatism results. This could be alleviated by employing nonlinear terms in $\tilde{k}x$ in Eq. (41), but we have not found that to be necessary. For future reference, we record that $A(2,1,0.9)=-2.149717032479-5.826341918286i$, which agrees to 10 decimals with an independent computation.‡

‡S. R. Bland, private communication, Nov. 16, 1978.

IV. Application to Unsteady Flow in Ventilated Wind Tunnels

Consider the problem of unsteady flow in a slotted wall wind tunnel. The ideal slotted wall boundary condition[6,7] is

$$p+\lambda\frac{\partial p}{\partial z}=0,\quad z=\eta \tag{42}$$

It is well known[3] that the resulting boundary value problem can be converted to an integral equation between the downwash w and the lifting pressure jump Δp,

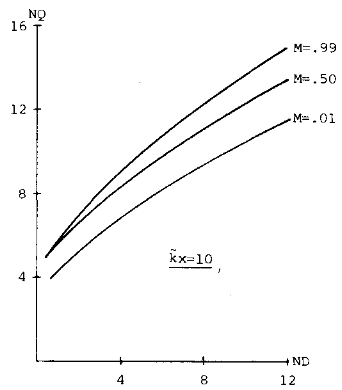
$$w(x)=\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}A(x-\xi,M,k,\eta,\lambda)\psi(\xi)\mathrm{d}\xi \tag{43}$$

where the unknown ψ is related to Δp by

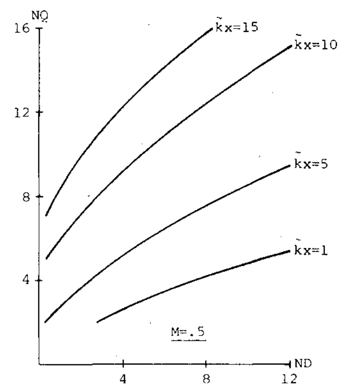
$$\Delta p(x)=\frac{4}{\beta}\sqrt{\frac{1-x}{1+x}}\psi(x) \tag{44}$$

For ease of reference, the airfoil integral equation (43) can be written in operator notation as

$$w=A\psi \tag{45}$$

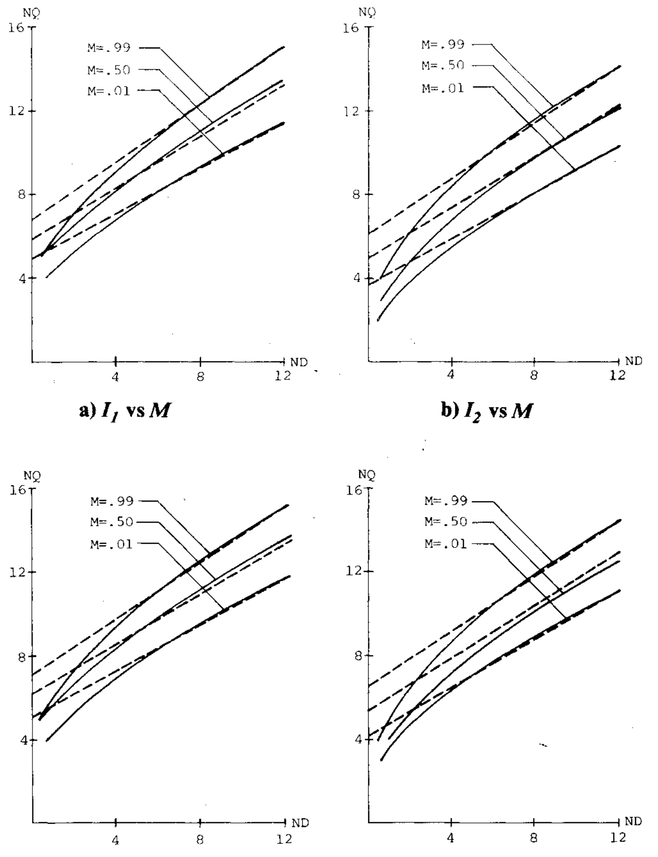


a) Effect of Mach number



b) Effect of frequency and distance

Fig. 2 Convergence to I_1 vs number of quadrature nodes.



a) I_1 vs M b) I_2 vs M

c) I_3 vs M d) I_4 vs M

Fig. 3 Computational convergence of $I_1 - I_4$ for $\bar{k}x = 10$.

Let

$$\alpha_n(x) = \frac{1}{\sqrt{\pi}} \frac{\cos[(n-\frac{1}{2})\cos^{-1}x]}{\cos(\frac{1}{2}\cos^{-1}x)}$$

$$\gamma_n(x) = \frac{1}{\sqrt{\pi}} \frac{\sin[(n-\frac{1}{2})\cos^{-1}x]}{\sin(\frac{1}{2}\cos^{-1}x)} \tag{46}$$

denote the airfoil polynomials,[3,4,8] where α_n is the nth downwash polynomial and γ_n is the nth pressure polynomial, and let

$$\psi_N = \sum_{n=1}^{N} a_n \gamma_n \tag{47}$$

be an N term approximation to ψ. This leaves a residual error in downwash which must be made small in some specific sense such as collocation, complex least squares, or Galerkin's method. Among these, collocation is computationally the fastest. It results in a set of simultaneous linear algebraic equations

$$[C_{mn}]\{a_n\} = \{w(x_m)\} \tag{48}$$

to be solved for the a_n, where the x_m are N distinct collocation nodes and where

$$C_{mn} = A\gamma_n(x_m) \tag{49}$$

determines the collocation matrix. Accurate evaluation of the collocation matrices is crucial to obtaining accurate solutions, as we shall see.

In an extensive analysis, Bland[3] showed that

$$A(x,k,M,\eta,\lambda) = H(x) + K_L(x,k,M) + K_C(x,k,M,\eta,\lambda) \tag{50}$$

where

$$K_L(x,k,M) = -\frac{ik}{\beta^2}\log|x| \tag{51}$$

and where $K_c(x,k,M,\eta,\lambda)$ is a continuous (but not analytic) function in x. See Ref. 8 for the equation of K_C.

In Bland's collocation method, the airfoil operator is split into three parts:

$$\mathcal{A} = \mathcal{H} + \mathcal{K}_L + \mathcal{K}_C \tag{52}$$

where $\mathcal{H}$, $\mathcal{K}_L$, and $\mathcal{K}_C$ are the integral operators corresponding to H, K_L, and K_C respectively, and the quadrature nodes are chosen as the zeros of α_{N+1}. The first two terms in Eq. (52) are given exactly by the integral transforms

$$\mathcal{H}\gamma_n(x) = \alpha_n(x) \tag{53}$$

and

$$-\mathcal{K}_L\gamma_n(x) = -\frac{ik}{\beta^2}\begin{cases} \dfrac{\alpha_{n+1}(x)}{2n} - (\log 2 + \frac{1}{2})\alpha_n(x), & n=1 \\ \dfrac{\alpha_{n+1}(x)}{2n} - \dfrac{\alpha_n(x)}{2n(n-1)} - \dfrac{\alpha_{n-1}(x)}{2(n-1)}, & n\geq 2 \end{cases} \tag{54}$$

The contribution due to $\mathcal{K}_C$ is approximated by using N point Jacobi-Gaussian quadrature,

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)\mathrm{d}x = \sum_{n=1}^{N}\frac{1-\xi_n}{N+\frac{1}{2}}f(\xi_n) + E_N \tag{55}$$

where the nodes ξ_n are the zeros of the airfoil polynomial γ_{N+1}, and with the error E_N for analytic functions being proportional to the 2Nth derivative of the integrand at some point in the interval $(-1, 1)$. Bland's collocation method then proceeds by using

$$C_{mn} = \mathcal{H}\gamma_n(x_m) + \mathcal{K}_L\gamma_n(x_m) + \mathcal{K}_C\gamma_n(x_m) \tag{56}$$

where $\mathcal{K}_C\gamma_n(x_m)$ is approximated by Eq. (55), and is completed by solving Eq. (48) with the collocation matrix of Eq. (56).

Instead of splitting the kernel as in Eq. (52), consider splitting off the terms in the Possio kernel,

$$A(x,k,M,\eta,\lambda) = H(x) + K_1(x,k,M)\log|x| + K_2(x,k,M) + \Delta K(x,k,M,\eta,\lambda) \tag{57}$$

Physically, ΔK is the interference kernel due to the presence of wind tunnel walls and we assume it to be analytic. This leads us to split the airfoil operator according to

$$\mathcal{A} = \mathcal{H} + \mathcal{K}_1 + (\mathcal{K}_2 + \Delta\mathcal{K}) \tag{58}$$

where $\mathcal{K}_1$, $\mathcal{K}_2$, and $\Delta\mathcal{K}$ represent the integral operators corresponding to K_1 log, K_2, and ΔK respectively. To compute

$$\mathcal{K}_1\gamma_n(x) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}K_1(x-\xi,k,M)\log|x-\xi|\gamma_n(\xi)\mathrm{d}\xi \tag{59}$$

efficiently, decomposes the interval of integration into four subintervals $[-1,a]$, $[a,x]$, $[x,b]$, and $[b,1]$, where -1

$<a<x<b<1$, and use the identity

$$\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}\log|x-\xi|f(x,\xi)\,d\xi$$

$$=\sqrt{1+a}\int_0^1\frac{1}{\sqrt{u}}\,[\sqrt{1-\xi}\log(x-\xi)f(x,\xi)]\,du \quad \xi=-1+(1+a)u$$

$$-(x-a)\int_0^1\log\frac{1}{u}\left[\sqrt{\frac{1-\xi}{1+\xi}}f(x,\xi)\right]du \qquad \xi=x-(x-a)u$$

$$+(x-a)\log(x-a)\int_0^1\left[\sqrt{\frac{1-\xi}{1+\xi}}f(x,\xi)\right]du \quad \xi=x-(x-a)u$$

$$+(b-x)\log(b-x)\int_0^1\left[\sqrt{\frac{1-\xi}{1+\xi}}f(x,\xi)\right]du \quad \xi=x+(b-x)u$$

$$-(b-x)\int_0^1\log\frac{1}{u}\left[\sqrt{\frac{1-\xi}{1+\xi}}f(x,\xi)\right]du \qquad \xi=x+(b-x)u$$

$$+\sqrt{1-b}\int_0^1\frac{1}{\sqrt{u}}\left[\frac{1-\xi}{\sqrt{1+\xi}}\log(\xi-x)f(x,\xi)\right]du \quad \xi=1-(1-b)u \tag{60}$$

Each of the six integrals appearing in the right-hand side of Eq. (60) possesses an integrand which is the product of an analytic function shown in square brackets, multiplied possibly by a function representing an inverse square root or logarithmic singularity, and thus may be computed accurately and efficiently by an appropriate Gaussian quadrature rule.

Consequently, an improved collocation method ensues by computing the collocation matrix according to

$$C_{mn}=\mathcal{K}\gamma_n(x_m)+\mathcal{K}_1\gamma_n(x_m)+(\mathcal{K}_2+\Delta\mathcal{K})\gamma_n(x_m) \tag{61}$$

where $\mathcal{K}\gamma_n$ is given exactly by Eq. (53), and where $\mathcal{K}_1\gamma_u$ and $(\mathcal{K}_2+\Delta\mathcal{K})\gamma_n$ are computed using Eqs. (60) and (55), respectively.

Before presenting numerical comparisons of Bland's collocation method [Eq. (56)] with Eq. (61), we make two observations. First, we have succeeded in proving[9] theoretically that both collocation methods converge in mean square to the solution of Eq. (43) whenever the downwash function satisfies

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,|w(x)|^2dx<\infty \tag{62}$$

While such a mathematical guarantee of convergence for general square integrable downwashes is at least reassuring, a more practical consideration is the actual rate of convergence. In this regard we point out that, neglecting quadrature error, the methods of Galerkin, collocation, and complex least squares are computationally equivalent whenever the collocation nodes are chosen as the quadature nodes used for least squares or Galerkin's method (see Ref. 4 for a complete proof). Consequently, the airfoil polynomials enjoy a distinct advantage and they provide superior convergence characteristics as a result. Second, by comparing Eq. (50) with Eq. (57) we see that the function

$$K_C(x,k,M,\eta,\lambda)=\frac{ik}{\beta^2}\,[1-e^{-ikx}F_1(x,k,M)]\log|x|$$

$$+K_2(x,k,M)+\Delta K(x,k,M,\eta,\lambda) \tag{63}$$

while continuous, contains singular terms of order $x\log|x|$. Such terms will not be integrated accurately by the Jacobi-Gaussian rule [Eq. (55)] and will result in convergence delay due to quadrature error in the collocation matrix. On the other hand, by splitting the kernel according to Eq. (57), all singularities can be treated efficiently and Eq. (61) can be expected to provide a more accurate and efficient means for computing the collocation matrices.

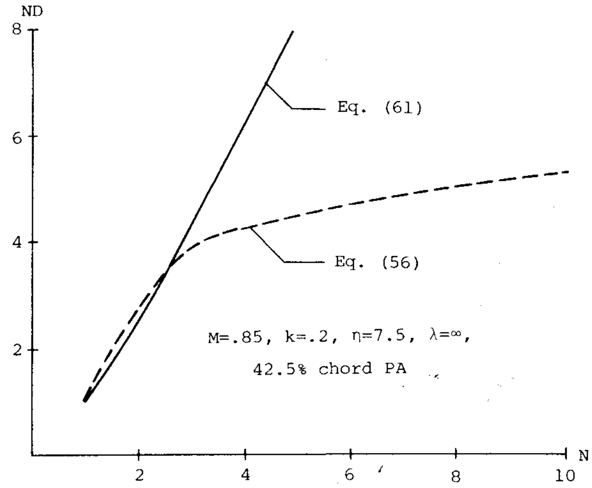


Fig. 4 Convergence to lift coefficient vs N.

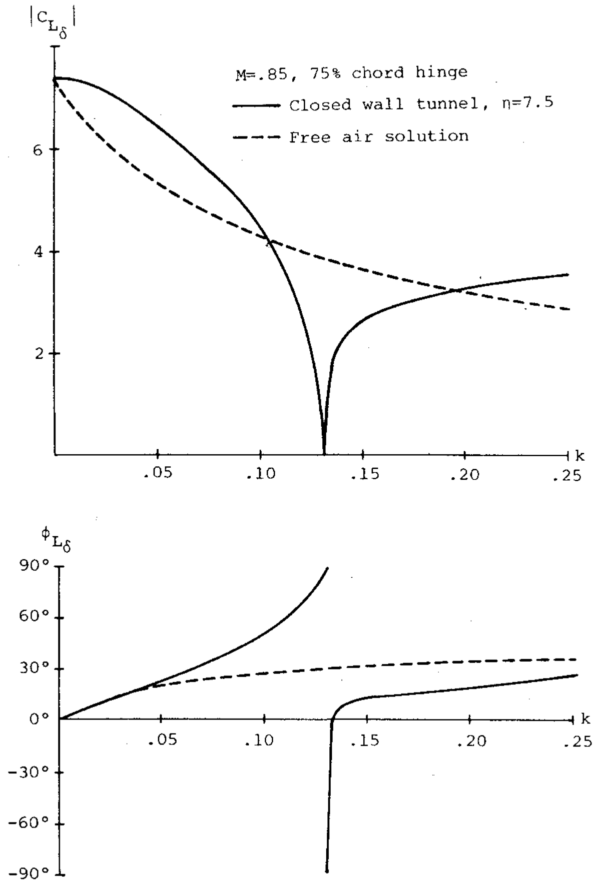


Fig. 5 C_{L_δ} vs k in wind tunnel and in free air.

Using the TWODI code,[4,8] the two collocation methods are compared in Fig. 4. The computational convergence with N of the lift coefficient is shown for an airfoil oscillating in pitch about the 42.5% chord in a closed wall wind tunnel. Both methods give comparable accuracy when only one or two pressure basis functions are used, producing about three decimal accuracy when $N=2$. However, beginning with $N=3$, the improved collocation method is considerably better, producing hundred-fold error reductions when $N\geq 4$.

The excellent convergence behavior of the improved collocation method is insensitive to ventilation coefficient and to higher aeroelastic downwash modes. This is discussed in Ref. 8. However when the airfoil has leading or trailing edge controls, the discontinuity in downwash at the hinge lines causes reduced convergence. This difficulty can be alleviated by the method of extrapolation,[8] but the effects of collocation error then become exacerbated by the presence of controls and the advantages of the improved collocation method become even more crucial in maintaining computational accuracy.

As a second application, consider the frequency response of an airfoil with a quarter chord trailing edge control in free air compared with that in a closed wall tunnel. Figure 5 indicates a significant difference between them in terms of magnitude as well as phase angle. The rate of change of magnitude with respect to frequency is not as large in the tunnel as it is in free air until the first resonant frequency is approached when it becomes much greater. Also, there is a rapid and discontinuous change in phase angle near the resonant frequency. Clearly, these interference effects would warrant careful consideration in the planning and interpretation of a wind tunnel test under such flow conditions.

V. Conclusions

The computational advantage of fully separating the singularities of generalized airfoil kernels has been demonstrated for unsteady flows in two-dimensional wind tunnels. This was made possible by a new formulation of the Possio kernel which explicitly identifies its singularities as well as those of more general kernels. An accurate method for computing the Possio kernel has been presented and optimization of the numerical technique has been described. These results for the free air Possio kernel have been used to split the generalized kernel for wind tunnel flows in a better way than before, resulting in a more accurate evaluation of the collocation matrices. This has enabled us to improve the collocation method so that an exponential rate of solution convergence is obtained. Numerical results indicate that six decimal accuracy requires only four terms in the pressure expansion.

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